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Axial anomaly with the overlap-Dirac operator in arbitrary dimensions

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Abstract: We evaluate for arbitrary even dimensions the classical continuum limit of the lattice axial anomaly defined by the overlap-Dirac operator. Our calculational scheme is simple and systematic. In particular, a powerful topological argument is utilized to determine the value of a lattice integral involved in the calculation. When the Dirac operator is free of species doubling, the classical continuum limit of the axial anomaly in various dimensions is combined into a form of the Chern character, as expected.

Keywords: Renormalization Regularization and Renormalons, Lattice Gauge Field Theories, Gauge Symmetry, Anomalies in Field and String Theories.

In this note, we evaluate for arbitrary even dimensions $d = 2n$ the classical continuum limit of the lattice axial anomaly defined by the overlap-Dirac operator [1]. This quantity has been studied many times [2]–[8]; but all treat $d = 4$ case. There also exist general arguments [3, 9, 7, 8] that the classical continuum limit reproduces the axial anomaly in the continuum theory, when the Dirac operator is free of species doubling. If it is possible, however, it is certainly preferable to demonstrate this by an explicit calculation. We do this for arbitrary dimensions and obtain the expected result; the classical continuum limit of the axial anomaly in various dimensions is combined into a form of the Chern character [10] [see eq. (41)]. Our calculation consists of two steps: First we determine $O(a^0)$ terms in the axial anomaly without expanding it with respect to the gauge coupling; this scheme is similar to that of refs. [11]–[14], [4, 5, 15]. Next, we evaluate a lattice integral involved; here a powerful topological argument is utilized to determine its value (closely related topological arguments can be found in refs. [16, 17, 2]). A combination of these two schemes provides a quite simple and systematic method to evaluate the classical continuum limit of the axial anomaly. To illustrate this point, we present a calculation of the axial anomaly with the Wilson-Dirac operator for arbitrary dimensions in appendix A.

The overlap-Dirac operator [1] is defined by[1]

$$D = \frac{1}{a}\left[1 - A(A^\dagger A)^{-1/2}\right], \qquad A = m_0 - aD_{\rm w}, \tag{1}$$

$$D_{\rm w} = \frac{1}{2}\left[\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - ar\nabla_\mu^*\nabla_\mu\right], \tag{2}$$

where m_0 and r are free parameters. We assume $r > 0$ in what follows. In the absence of the gauge field, the Dirac operator is free of species doubling if $0 < m_0/r < 2$. ∇_μ and ∇_μ^* in this expression are forward and backward covariant difference operators respectively[2]

$$\nabla_\mu\psi(x) = \frac{1}{a}\left[{\rm U}(x,\mu)\psi(x + a\hat\mu) - \psi(x)\right], \tag{3}$$

$$\nabla_\mu^*\psi(x) = \frac{1}{a}\left[\psi(x) - {\rm U}(x - a\hat\mu,\mu)^\dagger\psi(x - a\hat\mu)\right]. \tag{4}$$

By construction, the overlap-Dirac operator satisfies the Ginsparg-Wilson relation $\gamma_{d+1}D + D\gamma_{d+1} = aD\gamma_{d+1}D$ [18]. Due to this relation, the lattice fermion action $S_{\rm F} = a^d\sum_x \overline{\psi}(x) D\psi(x)$ is invariant under the modified chiral transformation [19]

$$\delta\psi(x) = \epsilon\gamma_{d+1}\left(1 - \frac{1}{2}aD\right)\psi(x), \qquad \delta\overline{\psi}(x) = \overline{\psi}\left(1 - \frac{1}{2}aD\right)\gamma_{d+1}\epsilon. \tag{5}$$

The fermion integration measure, however, acquires a non-trivial jacobian under the transformation [19]

$$\delta\prod_x {\rm d}\psi(x){\rm d}\overline{\psi}(x) = -2\left[a^d\sum_x \epsilon q(x)\right]\prod_x {\rm d}\psi(x){\rm d}\overline{\psi}(x), \tag{6}$$

[1] Our notations: a denotes the lattice spacing. Greek letters, μ, ν, ... run from 1 to $d = 2n$. Repeated indices are understood to be summed over, unless noted otherwise. $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_\mu^\dagger = \gamma_\mu$ and $\gamma_{d+1} = (-i)^n\gamma_1\cdots\gamma_d$; $\gamma_{d+1}^2 = 1$ and $\gamma_{d+1}^\dagger = \gamma_{d+1}$ follow from this.

[2] The link variable ${\rm U}(x,\mu)$ denotes a matrix in a (in general reducible) unitary representation of the gauge group, to which the fermion belongs.

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where

$$q(x)=\frac{1}{a^d}\,\mathrm{tr}\left[\gamma_{d+1}\left(1-\frac{1}{2}aD(x,x)\right)\right]. \tag{7}$$

This framework thus realizes a desired breaking pattern of the chiral Ward-Takahashi identity even with finite lattice spacings;[3] it is especially suitable for a study of phenomena associated to the axial anomaly, such as the U(1) problem [20]–[23]. The lattice axial anomaly $q(x)$ may also be regarded as the index density, because $a^d\sum_x q(x)$ is an integer depending on the gauge-field configuration [24].

We first note that a diagonal element of the kernel of an operator on the lattice can be expressed as

$$\begin{aligned}\mathcal{O}(x,x)&=\sum_y \mathcal{O}(x,y)\delta_{y,x}=\int_{-\pi}^{\pi}\frac{\mathrm{d}^dk}{(2\pi)^d}\,e^{-ikx/a}\sum_y \mathcal{O}(x,y)e^{iky/a}\\&=\int_{-\pi}^{\pi}\frac{\mathrm{d}^dk}{(2\pi)^d}\,e^{-ikx/a}\mathcal{O}e^{ikx/a}.\end{aligned} \tag{8}$$

So we note

$$\left\{\begin{matrix}A\\A^\dagger\end{matrix}\right\}e^{ikx/a}f(x)=e^{ikx/a}\left\{\mp\gamma_\mu(is_\mu+aQ_\mu)+m_0+r\left[\sum_\mu(c_\mu-1)+aR\right]\right\}f(x), \tag{9}$$

where

$$s_\mu=\sin k_\mu\,,\qquad c_\mu=\cos k_\mu\,, \tag{10}$$

and

$$Q_\mu=\frac{1}{2}\left(e^{ik_\mu}\nabla_\mu+e^{-ik_\mu}\nabla_\mu^*\right),\qquad R=\frac{1}{2}\sum_\mu\left(e^{ik_\mu}\nabla_\mu-e^{-ik_\mu}\nabla_\mu^*\right). \tag{11}$$

From eqs. (1) and (7), we thus have

$$\begin{aligned}q(x)=\frac{1}{2a^d}\int_{-\pi}^{\pi}\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{tr}\Bigg[\gamma_{d+1}&\left\{-\gamma_\mu(is_\mu+aQ_\mu)+m_0+r\left[\sum_\mu(c_\mu-1)+aR\right]\right\}\times\\&\times\left(-(is_\nu+aQ_\nu)^2+\left\{m_0+r\left[\sum_\nu(c_\nu-1)+aR\right]\right\}^2-\right.\\&\left.-\frac{a^2}{4}[\gamma_\nu,\gamma_\rho][Q_\nu,Q_\rho]+a^2r[\gamma_\nu Q_\nu,R]\right)^{-1/2}\Bigg]1,\end{aligned} \tag{12}$$

where we have used $\mathrm{tr}\,\gamma_{d+1}=0$ and $\gamma_\nu\gamma_\rho=\delta_{\nu\rho}+[\gamma_\nu,\gamma_\rho]/2$. In this form, an expansion of $q(x)$ with respect to the lattice spacing a is straightforward. In the classical continuum limit, the gauge potential $A_\mu(x)$ introduced by

$$\mathrm{U}(x,\mu)=\mathcal{P}\exp\left[a\int_0^1\mathrm{d}t\,A_\mu(x+(1-t)a\hat\mu)\right], \tag{13}$$

[3]Note that the jacobian is unity for flavor non-singlet chiral transformations for which $\mathrm{tr}\,\epsilon=0$.

is regarded as a smooth function of x of the order $O(a^0)$. The operators Q_μ and R are of $O(a^0)$ in this limit:

$$Q_\mu = c_\mu(\partial_\mu + A_\mu) + O(a)\,, \qquad \text{no sum over } \mu, \tag{14}$$

$$R = \sum_\mu i s_\mu(\partial_\mu + A_\mu) + O(a)\,, \tag{15}$$

and

$$[Q_\mu, Q_\nu] = c_\mu c_\nu F_{\mu\nu} + O(a)\,, \qquad \text{no sum over } \mu \text{ and } \nu\,, \tag{16}$$

$$[Q_\mu, R] = i c_\mu \sum_\nu s_\nu F_{\mu\nu} + O(a)\,, \qquad \text{no sum over } \mu\,, \tag{17}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Thus it is straightforward to find $O(a^0)$ terms in eq. (12), because the trace over Dirac indices requires at least d gamma matrices:

$$\mathrm{tr}\, \gamma_{d+1}\gamma_{\mu_1}\gamma_{\nu_1}\cdots\gamma_{\mu_n}\gamma_{\nu_n} = i^n 2^n \epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n}\,. \tag{18}$$

Namely, we have

$$\begin{aligned} q(x) = {} & \frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}^d k}{(2\pi)^d}\binom{-1/2}{n}\left\{\sum_\rho s_\rho^2 + \left[m_0 + r\sum_\rho (c_\rho - 1)\right]^2\right\}^{-n-1/2} \times \\ & \times\left(\mathrm{tr}\left\{\gamma_{d+1}\left[m_0 + r\sum_\sigma (c_\sigma - 1)\right]\left(-\frac{1}{2}\gamma_\mu\gamma_\nu c_\mu c_\nu F_{\mu\nu}\right)^n\right\} + \right. \\ & + \mathrm{tr}\left\{\gamma_{d+1}\gamma_\sigma s_\sigma\left[\left(-\frac{1}{2}\gamma_\mu\gamma_\nu c_\mu c_\nu F_{\mu\nu}\right)^{n-1} r\gamma_\lambda c_\lambda s_\tau F_{\lambda\tau} + \right.\right. \\ & \left.\left.\left. + \text{permutations}\right]\right\}\right) + O(a). \end{aligned} \tag{19}$$

A little calculation using the identity (18) shows

$$q(x) = \frac{(-i)^n}{2(2\pi)^d}\binom{-1/2}{n} I(m_0, r)\, \epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n}\, \mathrm{tr}\,(F_{\mu_1\nu_1}\cdots F_{\mu_n\nu_n}) + O(a)\,, \tag{20}$$

where the lattice integral $I(m_0, r)$ is given by

$$I(m_0, r) = \int_{\mathcal{B}} \mathrm{d}^d k\, \mathcal{I}(k; m_0, r)\,, \tag{21}$$

with

$$\begin{aligned} \mathcal{I}(k; m_0, r) = {} & \left(\prod_\mu c_\mu\right)\left\{\sum_\nu s_\nu^2 + \left[m_0 + r\sum_\nu (c_\nu - 1)\right]^2\right\}^{-n-1/2} \times \\ & \times\left[m_0 + r\sum_\rho (c_\rho - 1) + r\sum_\rho \frac{s_\rho^2}{c_\rho}\right], \end{aligned} \tag{22}$$

and

$$\mathcal{B} = \left\{ k_\mu \in \mathbb{R}^d \mid -\frac{\pi}{2} \le k_\mu \le \frac{3\pi}{2} \right\} . \tag{23}$$

Here we have shifted the integration region as $k_\mu \in [-\pi, \pi] \to k_\mu \in [-\pi/2, 3\pi/2]$ for later convenience.

We now turn to a calculation of the lattice integral $I(m_0, r)$. We utilize the following topological argument. We first introduce a mapping from the Brillouin zone $\mathcal{B}$ to the unit sphere S^d. The mapping is defined by

$$\theta_0 = m_0 + r \sum_\mu (c_\mu - 1) \, , \tag{24}$$

$$\theta_\mu = s_\mu \, , \qquad \text{for} \quad \mu = 1, \dots, d \, , \tag{25}$$

and

$$x_A = \frac{\theta_A}{\epsilon} \, , \qquad \epsilon = \sqrt{\sum_A \theta_A^2} \, , \tag{26}$$

where x_A $(A = 0, 1, \dots, d)$ is the coordinate of $\mathbb{R}^{d+1}$ in which the unit sphere $\sum_A x_A^2 = 1$ is embedded. Since this mapping is periodic on the Brillouin zone $\mathcal{B}$, we can regard $\mathcal{B}$ as a torus T^d. Namely $k_\mu \to x_A$ defines a mapping $f : T^d \to S^d$. The crucial observation is that the volume form on this sphere coincides with the integrand of $I(m_0, r)$:

$$\begin{aligned} \Omega &= \frac{1}{d!} \, \epsilon_{A_0 \cdots A_d} x_{A_0} \mathrm{d}x_{A_1} \wedge \cdots \wedge \mathrm{d}x_{A_d} \\ &= \frac{1}{d!} \frac{1}{\epsilon^{d+1}} \, \epsilon_{A_0 \cdots A_d} \theta_{A_0} \mathrm{d}\theta_{A_1} \wedge \cdots \wedge \mathrm{d}\theta_{A_d} \\ &= \mathcal{I}(k; m_0, r) \, \mathrm{d}k_1 \wedge \cdots \wedge \mathrm{d}k_d \, . \end{aligned} \tag{27}$$

This shows that the integral of Ω on a (sufficiently small) coordinate patch U on S^d is given by

$$\int_U \Omega = \mathrm{sgn} \left[\mathcal{I}(k^j; m_0, r) \right] \int_{U^j} \mathcal{I}(k; m_0, r) \, \mathrm{d}k_1 \wedge \cdots \wedge \mathrm{d}k_d \, , \tag{28}$$

where U^j $(j = 1, \dots, m)$ is a component of the inverse image of U under f, $f^{-1}(U)$:

$$f^{-1}(U) = U^1 \cup \cdots \cup U^m \subset T^d \, , \tag{29}$$

and k^j $(j = 1, \dots, m)$ is a certain point on U^j. For a sufficiently small U, U_j are pairwise disjoint. We take preimages of a point $y \in U$ under f, $f^{-1}(y)$, as k^j. Then by summing both sides of eq. (28) over j, we have[4]

$$\sum_j \int_{U^j} \mathcal{I}(k; m_0, r) \, \mathrm{d}k_1 \wedge \cdots \wedge \mathrm{d}k_d = (\deg f) \int_U \Omega \, , \tag{30}$$

[4]In deriving this relation, we have assumed that U is within the range of f, i.e., the inverse image $f^{-1}(U)$ is not empty. This relation itself, however, is meaningful even if U is not within the range of f, if one sets $\deg f = 0$ for such case. As a consequence, eq. (32) holds even if $f : T^d \to S^d$ is not a surjection, i.e., not an onto-mapping.

where the degree of the mapping f is given by

$$\deg f = \sum_{f(k^j)=y} \operatorname{sgn}\left[\mathcal{I}(k^j; m_0, r)\right] . \tag{31}$$

In general, the degree of the mapping $f : T^d \to S^d$ is defined by a sum of the signature of jacobian of the coordinate transformation between T^d and S^d over preimages of a point $y \in S^d$. An important mathematical fact [25] is that the degree takes the same value for all (regular) points y on S^d. Thus, by summing eq. (30) over all coordinate patches U of S^d, we have

$$I(m_0, r) = \int_{T^d} \mathcal{I}(k; m_0, r)\, \mathrm{d}k_1 \wedge \cdots \wedge \mathrm{d}k_d = (\deg f) \int_{S^d} \Omega \,, \tag{32}$$

where $\int_{S^d} \Omega$ is given by the volume of the unit sphere S^d:

$$\int_{S^d} \Omega = \mathrm{vol}(S^d) = \frac{2^{d+1}\pi^n n!}{d!} . \tag{33}$$

We may choose any point y on S^d to evaluate the degree (31). We choose $y = (1, 0, \ldots, 0)$. This requires

$$x_\mu(k^j) = \frac{s_\mu}{\epsilon} = 0 \,, \qquad \text{for} \quad \mu = 1, \ldots, d \,, \tag{34}$$

and

$$x_0(k^j) = \frac{m_0 + r\sum_\mu (c_\mu - 1)}{\left|m_0 + r\sum_\mu (c_\mu - 1)\right|} = 1 . \tag{35}$$

Note that eq. (35) is equivalent to the condition $m_0/r + \sum_\mu (c_\mu - 1) > 0$. Now eq. (34) implies that $k^j_\mu = 0$ or π for each direction μ. We denote the number of π's appearing in k^j by an integer $n_\pi \geq 0$,

$$k^j = (\underbrace{\pi, \ldots, \pi}_{n_\pi}, 0, \ldots, 0) \,, \tag{36}$$

irrespective of the position of π's. For a given n_π, the number of such k^j is $\binom{d}{n_\pi}$. The second relation (35) on the other hand requires $n_\pi < m_0/(2r)$. At those k^j, we have

$$\operatorname{sgn}\left[\mathcal{I}(k^j; m_0, r)\right] = \prod_\mu c_\mu = (-1)^{n_\pi} . \tag{37}$$

Thus eq. (31) gives

$$\deg f = \sum_{n_\pi=0}^{[m_0/2r]} \binom{d}{n_\pi} (-1)^{n_\pi} . \tag{38}$$

Combining eqs. (20), (32), (33) and (38), we finally obtain

$$q(x) = \frac{i^n}{(4\pi)^n n!} \sum_{n_\pi=0}^{[m_0/2r]} (-1)^{n_\pi} \binom{d}{n_\pi} \epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n} \operatorname{tr}\left(F_{\mu_1\nu_1} \cdots F_{\mu_n\nu_n}\right) + O(a) \,, \tag{39}$$

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where we have used

$$\binom{-1/2}{n} = \frac{1}{n!}\left(-\frac{1}{2}\right)^n (d-1)!!\,. \tag{40}$$

In particular, when $0 < m_0/r < 2$ with which the (free) overlap-Dirac operator is free of species doubling [1], eq. (39) reproduces the expected result. Multiplying the volume form $\mathrm{d}^d x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$ to $q(x)$, we have

$$q(x)\,\mathrm{d}^d x = \operatorname{tr}\exp\left(\frac{i}{2\pi}F\right)\bigg|_{\mathrm{d}^d x} + O(a), \tag{41}$$

where the field strength 2-form F is defined by $F = F_{\mu\nu}\,\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu/2$. This is our main result. When combined with cohomological arguments, it provides an absolute normalization of the lattice axial anomaly even with finite lattice spacings [26, 27] and with finite volumes [28].

A. Axial anomaly with the Wilson-Dirac operator

With the Wilson-Dirac operator, the chiral symmetry is explicitly broken by the Wilson term and the axial anomaly is regarded as arising from this explicit breaking [29]. The axial anomaly thus has a structure of a fermion one-loop diagram containing the Wilson term and, in our notation, it reads

$$q_{\mathrm{w}}(x) = -\frac{1}{a^d}\operatorname{tr}\left(\gamma_{d+1}\frac{a^2}{2}r\nabla_\mu^*\nabla_\mu\frac{1}{A}\right) = -\frac{1}{a^d}\operatorname{tr}\left[\gamma_{d+1}\frac{a^2}{2}r\nabla_\mu^*\nabla_\mu(A^\dagger A)^{-1}A^\dagger\right], \tag{42}$$

where the parameter m_0 in A of eq. (1) is replaced by am with m being the bare mass of the fermion. An expansion with respect to the lattice spacing is almost identical to the case of the overlap-Dirac operator in the text and we have

$$q_{\mathrm{w}}(x) = -\frac{i^n}{(2\pi)^d}\,I_{\mathrm{w}}(r)\,\epsilon_{\mu_1\nu_1\cdots\mu_n\nu_n}\operatorname{tr}\left(F_{\mu_1\nu_1}\cdots F_{\mu_n\nu_n}\right) + O(a), \tag{43}$$

where $I_{\mathrm{w}}(r) = \int_{\mathcal{B}} \mathrm{d}^d k\,\mathcal{I}_{\mathrm{w}}(k;r)$ and

$$\begin{aligned}\mathcal{I}_{\mathrm{w}}(k;r) = {}& \left(\prod_\mu c_\mu\right)\left\{\sum_\nu s_\nu^2 + r^2\left[\sum_\nu (c_\nu - 1)\right]^2\right\}^{-n-1} \\ & \times\left\{r^2\left[\sum_\rho (c_\rho - 1)\right]^2 + r^2\sum_\rho (c_\rho - 1)\sum_\sigma \frac{s_\sigma^2}{c_\sigma}\right\},\end{aligned} \tag{44}$$

which is independent of the fermion mass. To evaluate this lattice integral, we again utilize the mapping (24)–(26) with $m_0 = 0$. The mapping $k_\mu \to x_\mu$ (we do *not* include x_0 here) defines a mapping $g : \mathcal{B} \to B^d$, where B^d is the d dimensional ball, $B^d = \{x_\mu \in \mathbb{R}^d \mid \sum_\mu x_\mu^2 \leq 1\}$. More precisely, we have to regard all points of the boundary of the ball, $\sum_\mu x_\mu^2 = 1$, are identified, because $k = (0,\ldots,0)$ is mapped to the boundary of the ball. The crucial relation this time is that the volume form of the ball, $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$, coincides with the integrand of $I_{\mathrm{w}}(r)$ as one can verify. These observations show

$$I_{\mathrm{w}}(r) = (\deg g)\int_{B^d} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d = (\deg g)\,\mathrm{vol}(B^d)\,, \tag{45}$$

where the volume of the d dimensional ball, $\mathrm{vol}(B^d)$, is given by $\mathrm{vol}(B^d) = \pi^n/n!$ and the degree is

$$\deg g = \sum_{g(k^j)=y} \mathrm{sgn}\left[\mathcal{I}_{\mathrm{w}}(k^j; r)\right]. \tag{46}$$

As the point y on B^d, we choose the origin $y = (0, \ldots, 0)$. All momenta whose components k_μ are either 0 or π, *except* $k = (0, \ldots, 0)$, are mapped to $y = (0, \ldots, 0)$. Thus the degree is given by

$$\deg g = \sum_{n_\pi=1}^{d} \binom{d}{n_\pi} (-1)^{n_\pi} = -1. \tag{47}$$

Combining eqs. (43), (45) and (47), we have $q_{\mathrm{w}}(x)\,\mathrm{d}^d x = \mathrm{tr}[\exp iF/(2\pi)]|_{\mathrm{d}^d x} + O(a)$ for arbitrary Wilson parameter $r \neq 0$.

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